AD 607716

THE RAMO-WOOLDRIDGE CORPORATION
Guided Missile Research Division

THE RESPONSE OF MECHANICAL SYSTEMS TO RANDOM EXCITATION

by

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Report No. AM 5-13

23 November 1955

Aeromechanics Section

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Introduction

Mechanical systems are not always excited by a harmonic force of fixed frequency and amplitude. Often the excitation input is of random nature, and the response of the system displays no orderly trends. Instantaneous values and phase are meaningless in such cases, and the problem must be treated from a statistical approach. It is the purpose of this paper to outline such an approach as related to the dynamic response of structures.

The mathematical basis of the statistical technique has been exhaustively treated by Rice.[1] Engineering applications of statistical concepts have been followed with notable advances in fields such as communication. Applications to the structural field have only recently received attention, with treatment of problems of naturally statistical nature such as buffeting[2] and fatigue.[3] With greater utilization of rocket and jet engines, the statistical approach to structural dynamics assumes a role of increasing importance.

Two problems of interest to missile design have been treated here. The first problem deals with the longitudinal vibration of a slender rod excited by a random force at one end. The second problem is that of flexural vibration of a free-free beam excited by a random transverse force at one end. Briefly, the problem requires statistical description of the random excitation and determination of the frequency response of the structure including structural damping. With this information, it is possible to determine for any point in the structure the probability of exceeding any specified response such as stress, deflection, moment, or other quantities of interest.

Fundamental Concepts

To establish certain fundamental concepts, we will consider a linear system of single degree of freedom excited first by a harmonic force $P \cos \omega t$. The differential equation for such a system may be written as

$$\ddot{y} + 2\zeta \omega_n \dot{y} + \omega_n^2 y = \frac{P}{m} e^{i\omega t} \tag{1}$$

where only the real part of this equation is considered. Its steady-state solution is the real part of the equation,

$$y = Y e^{i(\omega t - \phi)}, \tag{2}$$

which upon substitution into equation (1) leads to the well-known results

$$y = \frac{P_o \cos(\omega t - \phi)}{k\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}} = \frac{P \cos(\omega t - \phi)}{|Z(\omega)|} \tag{3}$$

$$\phi = \tan^{-1} \frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}. \tag{4}$$

The quantity $Z(\omega)$ of equation (3) is the impedance function defined as the ratio of the input to the output. Impedance in a mechanical system is usually defined as the input force divided by the output velocity; however, for this discussion, we will apply the concept in a more general sense of input divided by the output without designating the quantities involved.

Of interest here is the mean square-response defined by the equation

$$\overline{y^2} = \frac{1}{T}\int_0^T y^2\, dt = \frac{\frac{1}{2}P^2}{|Z(\omega)|^2} = \frac{\overline{F^2}}{|Z(\omega)|^2}, \tag{5}$$

If now the frequencies ω_1, ω_2, ... differ only by a small amount, we approach a continuous spectrum, and the mean square-response in the frequency interval $\Delta\omega$ becomes

$$\overline{\Delta y^2} = \frac{\overline{\Delta F^2}}{|Z(\omega)|^2} = \frac{f(\omega)\,\Delta\omega}{|Z(\omega)|^2}. \tag{12}$$

Hence, by comparison with equation (9), we arrive at the result

$$g(\omega) = \frac{f(\omega)}{|Z(\omega)|^2} \tag{13}$$

$$\overline{y^2} = \int_0^{\infty} \frac{f(\omega)}{|Z(\omega)|^2}\, d\omega. \tag{14}$$

Equation (14) is the general equation for the mean-square response of the structure in terms of the power spectral density of the input and the impedance function of the system.

Evaluation of $\overline{y^2}$ for Systems with Small Damping

If the damping, ζ, is small, $Z(\omega)$ will undergo a large change near the resonant frequency, ω_n; and, if the variation in $f(\omega)$ is of lesser extent in this neighborhood, equation (14) can be approximated with good accuracy by the expression

$$\overline{y^2} = f(\omega_n) \int_{\omega_n - \Delta\omega}^{\omega_n + \Delta\omega} \frac{d\omega}{|Z(\omega)|^2} \tag{15}$$

To gain some useful concepts regarding this expression, we will consider the single-degree-of-freedom system of the previous section where, omitting the factor k^2, the impedance equation is

$$\frac{1}{|Z(\omega)|^2} = \frac{1}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2} \tag{16}$$

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A plot of equation (16), shown in Figure 1, indicates its peak value to be $1/4\zeta^2$ at $\omega/\omega_n = 1$. It can be shown by solving the equation

$$\frac{1}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2} = \frac{1}{2}\left(\frac{1}{4\zeta^2}\right) \tag{17}$$

for small ζ, that the frequencies at the half-power level are

$$\frac{\omega}{\omega_n} = 1 \pm \zeta, \tag{18}$$

which determines the width of the $1/|Z(\omega)|^2$ curve at this point to be 2ζ.

The area under the $1/|Z(\omega)|^2$ curve for small ζ can be evaluated by letting $\omega/\omega_n = 1 + \epsilon$, where $\epsilon << 1$. The integral of equation (15) for small ζ then becomes

$$I = \frac{\omega_n}{4}\int_{-\epsilon}^{\epsilon} \frac{d\epsilon}{\epsilon^2 + \zeta^2} = \frac{\omega_n}{4\zeta} \tan^{-1} \frac{\epsilon}{\zeta}. \tag{19}$$

Since ζ is assumed small, ϵ can be several times ζ and still remain small compared to unity, and hence this integral is given with sufficient accuracy by

$$I = \frac{\pi}{4\zeta} \omega_n. \tag{20}$$

The mean-square response then becomes

$$\overline{y^2} = \frac{\pi \omega_n}{4\zeta k^2} f(\omega_n). \tag{21}$$

It should be noted here that the integrated area is equal to $\pi/2$ times the peak value times the width at the half-power point, a relationship which will be of use later.

Continuous and Multidegree Freedom Systems

Although equation (21) was derived from $Z(\omega)$ associated with a system of single degree of freedom, it can be shown to be applicable to continuous and multidegree freedom systems with slight modification. In such cases, $1/|Z(\omega)|^2$ has many peaks; however, the integral $\int_0^\infty d\omega/|Z(\omega)|^2$ at each peak has the same form as equation (19) except for a term which is a function of position x in the system and which is not involved in the evaluation of the integral.

The impedance function in the neighborhood of resonance can here be expressed in the form

$$\frac{1}{|Z(\omega)|^2} \cong \frac{K(x, \omega_n)}{\epsilon^2 + r^2} \tag{22}$$

where $K(x, \omega_n)/r^2$ is the peak value and r is the term associated with damping. Again the integral $\int_0^\infty d\omega/|Z(\omega)|^2$ can be evaluated by the rule of the previous section, and the mean-square response for such systems take the form

$$\overline{y^2} = \frac{\pi}{r} \sum_n \omega_n \; K(x, \; \omega_n) \; f(\omega_n). \tag{23}$$

Illustrative examples in the following sections will clarify this equation.

Longitudinal Motion of Slender Rods under Random Excitation

We will consider here a slender rod excited axially at the end, $x = 0$, by a random force, $F(t)$, and free at the other end, $x = \ell$. Structural damping will be included by a complex stiffness, $E(1 + ig)$. The differential equation of motion for an element of the bar is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2 (1 + ig)} \frac{\partial^2 u}{\partial t^2}, \tag{24}$$

where

$$c = \sqrt{\frac{AE}{m}} = \text{velocity of the compressional wave}$$

$$g = \text{percentage of structural damping (in decimals).}$$

Taking the Laplace transformation and making the subsitution

$$(1 + ig)^{-1} \cong (1 - ig),$$

the subsidiary equation becomes,

$$\frac{d^2 \bar{u}(x, s)}{dx^2} = \left(\frac{s}{c}\right)^2 (1 - ig)\, \bar{u}(x, s). \tag{25}$$

The general solution for the above equation is

$$\bar{u}(x, s) = C_1 e^{s/c \sqrt{1 - ig}\; x} + C_2 e^{-s/c \sqrt{1 - ig}\; x} \quad . \tag{26}$$

Fitting this solution to the boundary conditions,

$$\bar{\sigma}(o, s) = \frac{\bar{F}(o, s)}{A} = -E \frac{d\,\bar{u}(o, s)}{dx} \tag{27}$$

$$\bar{\sigma}(\ell, s) = 0, \tag{28}$$

we arrive at an expression for the subsidiary stress, $\bar{\sigma}(x, s)$, which is

$$\bar{\sigma}(x, s) = \bar{\sigma}(o, s) \frac{\sinh \frac{s\ell}{c} \sqrt{1 - ig} \left(\frac{x}{\ell} - 1\right)}{\sinh \frac{s\ell}{c} \sqrt{1 - ig}} \tag{29}$$

To determine $1/|Z(\omega)|^2$, we evaluate this equation for a harmonic excitation, $F(o, t) = F_o e^{i\omega t}$, which results in the expression

$$\sigma(x, t) = \left\{ \frac{\sinh \frac{i\omega\ell}{c} \sqrt{1 - ig} \left(\frac{x}{\ell} - 1\right)}{\sinh \frac{i\omega\ell}{c} \sqrt{1 - ig}} \right\} \sigma_o e^{i\omega t} \quad . \tag{30}$$

If we define the impedance, $|Z(\omega)|$, for this problem to be the ratio of the input stress to the output stress at any position x, then $1/|Z(\omega)|$ is given by the absolute value of the quantity in the braces in equation (30). Making the substitution $\sqrt{1 - ig} \cong 1 - i(g/2)$ where g is small, we obtain the desired result to be

$$\frac{1}{|Z(\omega)|^2} = \frac{\sin^2 \frac{\omega \ell}{c}\left(\frac{x}{\ell} - 1\right)}{\sin^2 \frac{\omega \ell}{c} + \left(\frac{g}{2}\right)^2 \left(\frac{\omega \ell}{c}\right)^2 \cos^2 \frac{\omega \ell}{c}} \tag{31}$$

We note in this expression that if the damping $g = 0$, then $1/|Z(\omega)|^2$ becomes infinite when

$$\frac{\omega \ell}{c} = \pi, 2\pi, 3\pi, \ldots, n\pi, \ldots,$$

thus identifying the natural frequencies of the bar to be

$$\omega_n = n\pi \frac{c}{\ell} . \tag{32}$$

Retaining g, the impedance function in the neighborhood of resonance becomes

$$\frac{1}{|Z(\omega)|^2} = \frac{\sin^2 n\pi\left(\frac{x}{\ell} - 1\right)}{\left(\frac{\omega_n \ell}{c}\right)^2 \epsilon^2 + \left(\frac{n\pi g}{2}\right)^2} = \frac{\sin^2 n\pi\left(\frac{x}{\ell} - 1\right)}{(n\pi)^2 \left[\epsilon^2 + \left(\frac{g}{2}\right)^2\right]} \tag{33}$$

which is in the form of equation (22), where $r = g/2$ and

$$K(x, \omega_n) = \frac{\sin^2 n\pi\left(\frac{x}{\ell} - 1\right)}{(n\pi)^2} .$$

Thus by substitution into equation (23), or by using the previous area rule, the mean-square response becomes

$$\overline{\sigma^2} = \frac{2\pi}{g} \sum_n \frac{\omega_n \sin^2 n\pi \left(\frac{x}{\ell} - 1\right)}{(n\pi)^2} f(\omega_n)$$

$$= \frac{2\pi}{g} \sum_n \frac{c}{n\pi\ell} \sin^2 n\pi \left(\frac{x}{\ell} - 1\right) f(\omega_n), \tag{34}$$

which can be evaluated when the power spectral density, $f(\omega)$, of the input stress is known.

Power Spectral Density of Excitation

The curve for the power spectral density, $f(\omega)$, may have various shapes depending on the nature of the source of excitation. However, from energy considerations, $f(\omega)$ for all forms of excitation must approach zero as ω approaches infinity. In all cases $f(\omega)$ may be defined by its mean-square value and some characteristic frequency, ω_c.

One variation of $f(\omega)$ frequently considered is the monotonically decreasing curve of Figure 2, defined by the equation

$$f(\omega) = \frac{2}{\sqrt{\pi}} \frac{\overline{F^2}}{\omega_c} e^{-(\omega/\omega_c)^2} . \tag{35}$$

It should be noted that the above equation satisfie the original definition

$$F^2 = \int_0^\infty f(\omega)\, d\omega.$$

The choice of the characteristic frequency, ω_c, is arbitrary; however, it is convenient to relate it to the frequency corresponding to the half-power point of the input spectral density. If $\omega_{1/2}$ represents this half-power point, then $\omega_c = 2.1\, \omega_{1/2}$, and 84% of the input energy is contained in the frequency range 0 to ω_c.

Another variation of $f(\omega)$ which is of interest is the rectangular distribution of Figure 3 with a cut-off frequency, ω_c. The spectral density, $f(\omega)$, is then defined by the equation

$$f(\omega) = \frac{\overline{F^2}}{\omega_c}, \qquad \omega > \omega_c$$

$$= 0, \qquad \omega > \omega_c. \tag{36}$$

This distribution, which is often referred to as "white noise," can sometimes be used as an equivalent spectrum for the more general types of distribution.

To establish certain concepts regarding the excitation spectrum, it is well to discuss how $f(\omega)$ is determined by experiment. The essential components of the measurement apparatus, shown in Figure 4, consists of a band-pass filter and a meter or indicator which will read the mean-square values. Equipment is commercially available where the central frequency of the band-pass filter is continually swept and the spectrum for $f(\omega)$ is displayed on an oscilloscope. With a continuous spectrum, the band of frequencies passed by the filter represents a sum of the harmonic oscillations of frequencies differing by increments, $\delta\omega$. Noting that

$$A \cos(\omega - \delta\omega)t + A \cos\left[(\omega + \delta\omega)t + \phi\right] = 2A \cos(\delta\omega t + \tfrac{1}{2}\phi) \cos(\omega t + \tfrac{1}{2}\phi) \tag{37}$$

it is evident that the result is a large number of amplitude-modulated oscillations approaching a random amplitude fluctuation at frequencies of order $\delta\omega$.

In interpreting equations (35) or (36), one must not assume that the height of the $f(\omega)$ curve diminishes with large values of the cut-off frequency, ω_c. The spectral analyzer in indicating $f(\omega) = \overline{\Delta F^2}/\Delta\omega$ is unaware of the excitation outside the instantaneous pass band $\Delta\omega$, and hence $f(\omega)$ is unaffected by the extent of the frequency range of the spectrum. Thus the proper interpretation of these equations is that $\overline{F^2}/\omega_c$ remains essentially constant or that the mean-square value of the excitation increases with ω_c.

Numerical Evaluation of $\overline{\sigma^2}$

To examine how the mean-square response, $\overline{\sigma^2}$, for the longitudinal motion of the rod is related to the mean-square value of the excitation stress, $\overline{\sigma_o^2}$, we will consider the two types of power spectral densities discussed in the previous section.

Substituting equation (35) into equation (34), the ratio of the mean-square values for the monotonic spectral density is expressed by the equation

$$\frac{\overline{\sigma^2}}{\overline{\sigma_o^2}} = \frac{4}{g\sqrt{\pi}}\left(\frac{c}{\omega_c \ell}\right)\sum_{n=1}^{\infty}\frac{1}{n}\, e^{-\left(n\pi \frac{c}{\omega_c \ell}\right)^2} \sin^2 n\pi\left(\frac{x}{\ell}-1\right). \qquad (38)$$

In a similar manner, the substitution of equation (36) for the rectangular spectrum, with an upper limit on the summation corresponding to the cut-off frequency, ω_c, or $n_c = 1/\pi(c/\omega_c \ell)$, leads to the result

$$\frac{\overline{\sigma^2}}{\overline{\sigma_o^2}} = \frac{2}{g}\left(\frac{c}{\omega_c \ell}\right)\sum_{n=1}^{n_c}\frac{1}{n}\sin^2 n\pi\left(\frac{x}{\ell}-1\right). \qquad (39)$$

It is evident from these equations that $\overline{\sigma^2}/\overline{\sigma_o^2}$ can be plotted as a function of the nondimensional quantity, $(c/\omega_c \ell)$, with x/ℓ and g as parameters.

Results of calculations carried out for $x/\ell = 1/2$ and $g = 0.01$ are plotted in Figure 5. These curves indicate that for small values of $(c/\omega_c \ell)$, the rectangular and the monotonic spectral densities result in nearly the same values of the mean-square ratios, which is not surprising when one compares the above two equations. Since g appears as a linear factor in these equations, results for other values of damping are obtainable from these curves by simple division.

Probability of Exceeding a Specified Response

For a random function such as σ, the normal probability distribution is a reasonable assumption. Letting λ be any random variable in question, the normal distribution curve shown in Figure 6 establishes the probability that λ will be found in the region λ to $\lambda + d\lambda$ to be

$$\frac{2}{\sqrt{\pi}} \quad e^{-\lambda^2} \quad d\lambda. \tag{40}$$

Of interest here is the question of the probability of σ exceeding some specified value, σ_1, which is n times the root-mean-square response, $\sqrt{\overline{\sigma^2}}$. To arrive at an answer to this question, we let $\lambda = \sigma/\sqrt{2\overline{\sigma^2}}$, in which case the probability that σ lies between σ and $\sigma + d\sigma$ is

$$\frac{2}{\sqrt{\pi}} \quad \frac{e^{-\frac{1}{2}\frac{\sigma^2}{\overline{\sigma^2}}}}{\sqrt{2\overline{\sigma^2}}} \quad d\sigma \tag{41}$$

Thus the probability of σ exceeding a certain value, $\sigma_1 = n\sqrt{\overline{\sigma^2}}$, is determined by integration to be

$$P(\sigma > \sigma_1) = \frac{2}{\sqrt{\pi}} \int_{\sigma_1}^{\infty} \frac{e^{-\frac{1}{2}\frac{\sigma^2}{\overline{\sigma^2}}}}{\sqrt{2\overline{\sigma^2}}} \, d\sigma = \text{erfc}\left(\frac{\sigma_1}{\sqrt{2\overline{\sigma^2}}}\right)$$

$$= \text{erfc}\left(\frac{1}{2}\, n\right) \tag{42}$$

As an example, if $c/\omega_c \ell = 0.08$, the mean-square ratio for the white-noise distribution is found from Figure 5 to be $\overline{\sigma^2}/\overline{\sigma_o^2} = 23$. Thus the root-mean-square ratio is $\sqrt{\overline{\sigma^2}} = 4.80 \sqrt{\overline{\sigma_o^2}}$. The probability of the stress exceeding some number, such as 3 times the root-mean-square stress at $x/\ell = 1/2$ and $g = 0.01$, or 3 times $4.80 \sqrt{\overline{\sigma_o^2}} = 14.4$ times the root-mean-square value of the input stress, is

$$P\left(\sigma > 3\sqrt{\overline{\sigma^2}}\right) = P\left(\sigma > 14.4\sqrt{\overline{\sigma_o^2}}\right) = \text{erfc}\ (1.50) = 0.034.$$

Lateral Motion of Beams Under Random Excitation

We consider next the lateral response of a beam excited at the end, $x = 0$, by a random force, $F(t)$, with the end, $x = \ell$, free. Again we include structural damping by a complex stiffness, $E(1 + ig)$, and write the differential equation for the beam element for harmonic excitation.

$$\frac{d^4y}{dx^4} - \frac{\omega^2 m}{(1 + ig)\,EI}\; y = 0. \tag{43}$$

Letting

$$\begin{aligned} \beta^4 &= \frac{m\omega^2}{EI} \\ (1 + ig)^{-1} &\cong 1 - ig \\ \overline{\beta^4} &= (1 - ig)\,\beta^4, \end{aligned} \tag{44}$$

the above equation becomes

$$\frac{d^4y}{dx^4} - \overline{\beta^4}\, y = 0. \tag{45}$$

We thus obtain the same solution as that of the undamped beam, except that β is now replaced by its complex counterpart, $\overline{\beta}$. We will, however, carry out the solution in terms of β and account for $\overline{\beta}$ at the very end.

Using the Laplace transformation with x as the original variable, it is possible to arrive at a general solution in terms of quantities at $x = 0$; and, by successive differentiation, equations for deflection, slope, moment and shear can be obtained and expressed in matrix form.[4]

$$\begin{bmatrix} y(x) \\ \frac{1}{\beta} y'(x) \\ \frac{1}{\beta^2} y''(x) \\ \frac{1}{\beta^3} y'''(x) \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{bmatrix} \begin{bmatrix} y(0) \\ \frac{1}{\beta} y'(0) \\ \frac{1}{\beta^2} y''(0) \\ \frac{1}{\beta^3} y'''(0) \end{bmatrix} \tag{46}$$

where

$$
\begin{aligned}
a &= \frac{1}{2}(\cosh \beta x + \cos \beta x) \\
b &= \frac{1}{2}(\sinh \beta x + \sin \beta x) \\
c &= \frac{1}{2}(\cosh \beta x - \cos \beta x) \\
d &= \frac{1}{2}(\sinh \beta x - \sin \beta x).
\end{aligned} \tag{47}
$$

When $x = \ell$, we have the boundary conditions $y''(\ell) = y'''(\ell) = 0$. At the end, $x = 0$, the boundary conditions are $y''(0) = 0$, and $1/\beta^3 \left[y'''(0)\right] = P_o/\beta^3 EI$, where the excitation is $P_o \cos \omega t$. With these conditions, the remaining constants become

$$
\begin{aligned}
y(0) &= \frac{-P_o}{\beta^3 EID} \begin{vmatrix} b & d \\ a & c \end{vmatrix}_{x=\ell} \\
\frac{1}{\beta} y'(0) &= \frac{-P_o}{\beta^3 EID} \begin{vmatrix} c & b \\ b & a \end{vmatrix}_{x=\ell} \\
D &= \begin{vmatrix} c & d \\ b & c \end{vmatrix}_{x=\ell} = -\frac{1}{2}(\cosh \beta\ell \cos \beta\ell - 1).
\end{aligned} \tag{48}
$$

It is now possible to write the equation for any quantity $y(x)$ to $y'''(x)$. For instance, the equation for $y''(x)$ becomes

$$
y''(x) = \frac{P_o}{\beta EI} \left\{ b(x) - \begin{vmatrix} b & d \\ a & c \end{vmatrix}_{x=\ell} \frac{c(x)}{D} - \begin{vmatrix} c & b \\ b & a \end{vmatrix}_{x=\ell} \frac{d(x)}{D} \right\} \tag{49}
$$

It is evident here that the natural frequencies are determined from the equation $D = 0$, in which case $y''(x) \to \infty$.

When damping is included, we replace β in the above solution by $\overline{\beta}$. However, since small damping is assumed and we are interested in the impedance function only in the neighborhood of resonance, we need to replace β by $\bar{\beta}$ only in terms which tend to zero, or in the expression for D.

Letting $\bar{\beta} \cong \beta\,(1 - ig)^{1/4} \cong \beta\,(1 - i\,\frac{g}{4})$, the expression for D becomes

$$D = (\cosh \beta\ell \cos \beta\ell - 1) + i\,\frac{g}{4}\,\beta\ell\,(\cosh \beta\ell \sin \beta\ell - \sinh \beta\ell \cos \beta\ell) \tag{50}$$

where terms containing g^2 have been omitted as negligible. We next replace $\beta\ell$ by $\beta_n\ell\,(1 + \epsilon)$ in the neighborhood of resonance; and again throwing out infinitesimals of higher order as well as

$$\cosh \beta_n\ell \cos \beta_n\ell - 1 = 0,$$

we arrive at the result

$$\begin{aligned} D &= (\sinh \beta_n\ell \cos \beta_n\ell - \cosh \beta_n\ell \sin \beta_n\ell)\,(\beta_n\ell)\,(\epsilon - i\,\tfrac{g}{4}) \\ &= \frac{1}{2}\begin{vmatrix} b & d \\ a & c \end{vmatrix}(\beta_n\ell)\,(\epsilon - i\,\tfrac{g}{4}). \end{aligned} \tag{51}$$

Substituting in this value of D and noting that in the neighborhood of resonance b(x) is negligible compared to the other two terms, the final equation for the moment becomes

$$M(x) = E I\, y''(x) = \frac{2 P_o \ell}{(\beta_n\ell)^2\,(\epsilon - i\,\frac{g}{4})}\left\{ c(x) + \frac{\begin{vmatrix} c & b \\ b & a \end{vmatrix}}{\begin{vmatrix} b & d \\ a & c \end{vmatrix}}\, d(x) \right\}. \tag{52}$$

If we now define the impedance as the ratio of the input moment, $M_o = P_o\ell$, to the output moment, M(x), the quantity of interest becomes

$$\frac{1}{|Z(\omega)|^2} = \frac{4\left\{ c(x) + \frac{\begin{vmatrix} c & b \\ b & a \end{vmatrix}}{\begin{vmatrix} b & d \\ a & c \end{vmatrix}}\, d(x) \right\}^2_{\beta\ell = \beta_n\ell}}{(\beta_n\ell)^4\left\{ \epsilon^2 + (\frac{g}{4})^2 \right\}}. \tag{53}$$

By the previous rule for integrated area, the mean-square moment contribution for mode k is given as

$$\overline{M_k^2} = \frac{\pi}{2}\left[2\left(\frac{g}{4}\right)\omega_k\right]\left[\frac{1}{|Z(\omega)|^2}\bigg|_{\epsilon = 0} f(\omega_k)\right] . \tag{54}$$

Thus, summing over all modes, we have

$$\overline{M^2} = 4\pi \sum_n \frac{\omega_n f(\omega_n)}{(\beta_n \ell)^4 \left(\frac{g}{4}\right)} \left\{ c(x) + \frac{\begin{vmatrix} c & b \\ b & a \end{vmatrix}}{\begin{vmatrix} b & d \\ a & c \end{vmatrix}} d(x) \right\}^2_{\beta\ell = \beta_n \ell} . \tag{55}$$

Since

$$\omega_n = (\beta_n \ell)^2 \sqrt{\frac{EI}{m\ell^4}}$$

and

$$\beta_n \ell \cong (2n + 1)\frac{\pi}{2},$$

this equation can also be written as

$$\overline{M^2} = \frac{64}{\pi g} \sum_n \frac{f(\omega_n)\sqrt{\frac{EI}{m\ell^4}}}{(2n + 1)^2} \left\{ c(x) + \frac{\begin{vmatrix} c & b \\ b & a \end{vmatrix}}{\begin{vmatrix} b & d \\ a & c \end{vmatrix}} d(x) \right\}^2_{\beta\ell = \beta_n \ell} . \tag{56}$$

The squared quantity within the braces can be identified as being equal to half the tabulated results, $1/2\ \phi(x)$, of Young and Felgar,[5] and for this problem corresponds to the case of the free-free beam.

Numerical Evaluation of M^2

As in the longitudinal case, we evaluate equation (56) for the two types of spectral densities defined by equations (35) and (36). For the monotonic variation of $f(\omega)$, the result reduces to

$$\frac{\overline{M^2}}{\overline{M_o^2}} = \frac{128}{g\,\pi^{3/2}}\,\frac{1}{\omega_c}\sqrt{\frac{EI}{m\ell^4}}\sum_{n=1}^{\infty}\frac{e^{-(2n+1)^4\left(\frac{\pi}{2}\right)^4\left(\frac{1}{\omega_c}\sqrt{\frac{EI}{m\ell^4}}\right)^2}}{(2n+1)^2}\left\{c(x)+\frac{\begin{vmatrix} c & b \\ b & a \end{vmatrix}}{\begin{vmatrix} b & d \\ a & c \end{vmatrix}}\,d(x)\right\}^2 . \quad (57)$$

For the white-noise spectral density, $f(\omega) = \overline{M_o^2}/\omega_c$, we obtain the ratio of the mean squares to be

$$\frac{\overline{M^2}}{\overline{M_o^2}} = \frac{64}{\pi g}\,\frac{1}{\omega_c}\sqrt{\frac{EI}{m\ell^4}}\sum_{n=1}^{n_c}\frac{1}{(2n+1)^2}\left\{c(x)+\frac{\begin{vmatrix} c & b \\ b & a \end{vmatrix}}{\begin{vmatrix} b & d \\ a & c \end{vmatrix}}\,d(x)\right\}^2 . \quad (58)$$

The summation in this case is terminated at the mode number, n_c, corresponding to the cut-off frequency, ω_c, this relationship being

$$\omega_c = (2n_c+1)^2\left(\frac{\pi}{2}\right)^2\sqrt{\frac{EI}{m\ell^4}} . \quad (59)$$

Equations (57) and (58) were numerically evaluated for $x/\ell = 1/2$ and $g = 0.01$ and plotted in Figure 7. These curves are to be interpreted in a manner similar to that of the longitudinal case, and the probability equation (42) is again applicable with n interpreted as the number of times the specified moment exceeds the root-mean-square moment.

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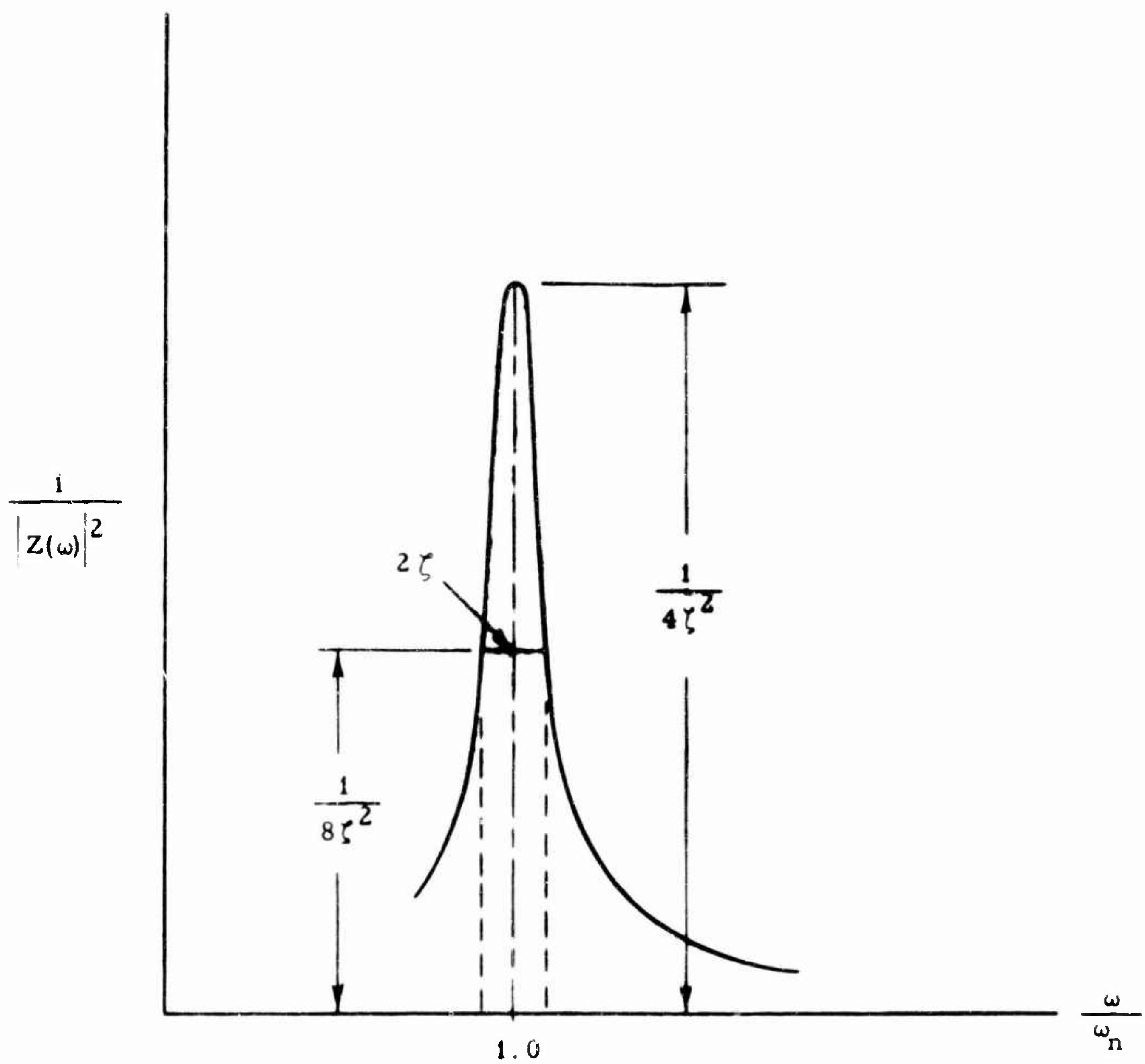

Fig. 1. Frequency Spectrum of System with Single Degree of Freedom.

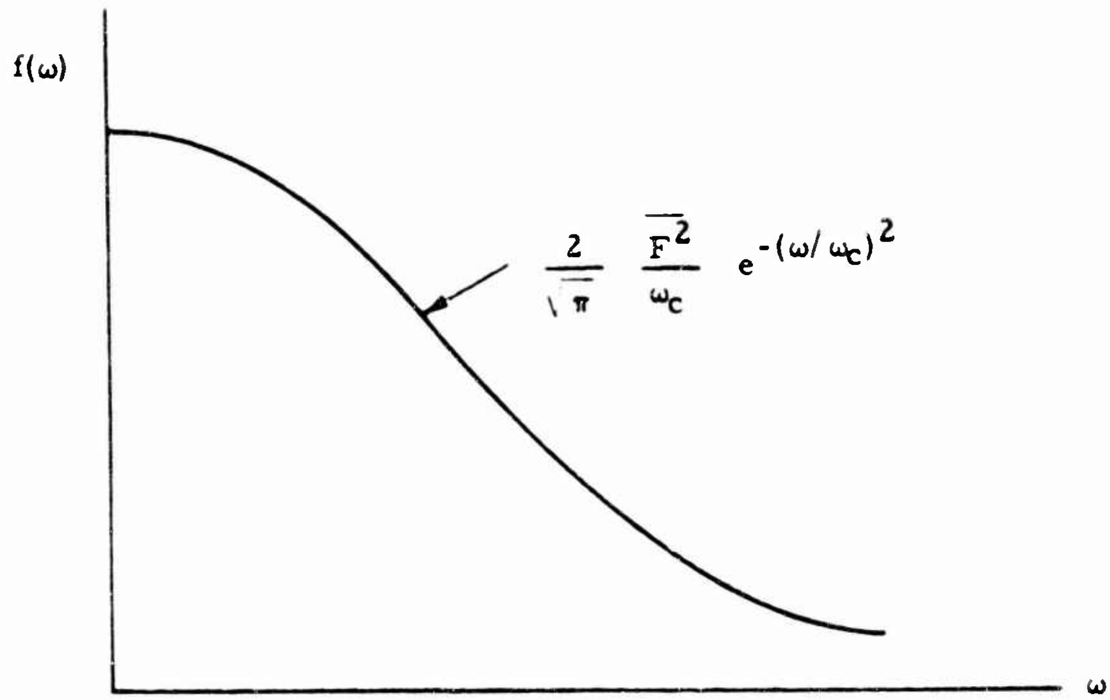

Fig. 2. Monotonically Decreasing Spectrum.

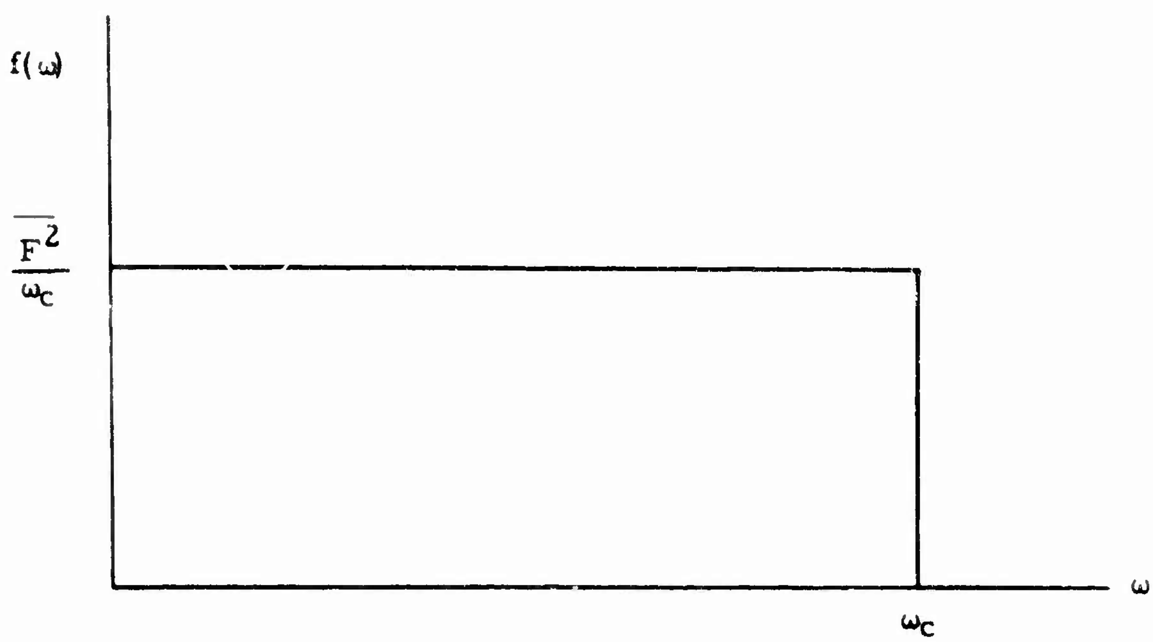

Fig. 3. "White Noise" Spectrum.

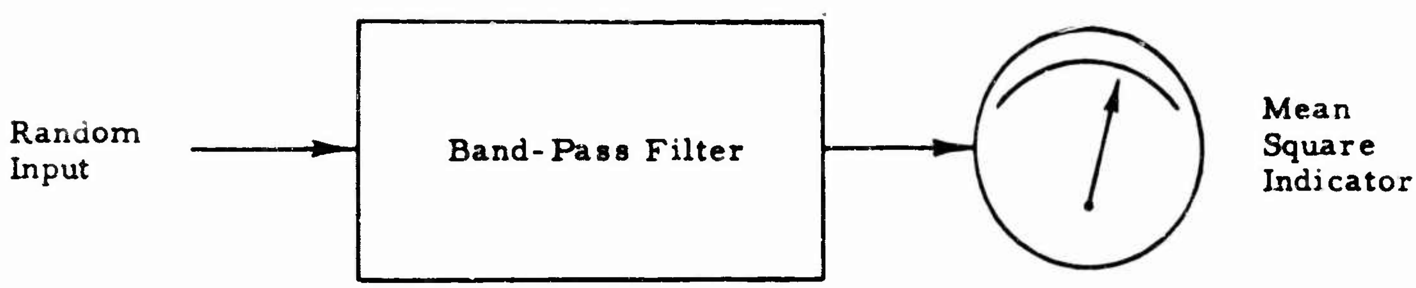

Fig. 4. Measurement Apparatus.

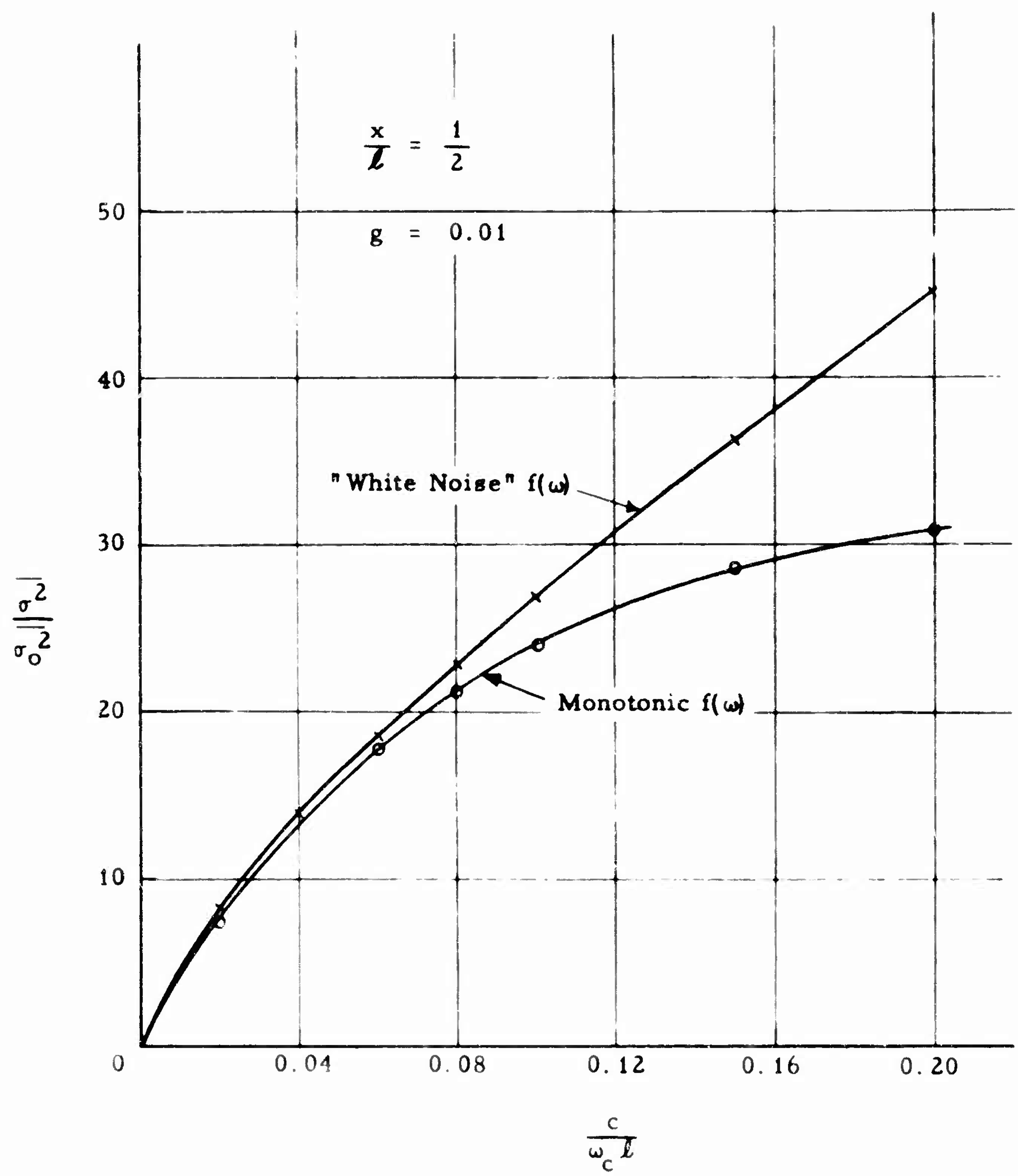

Fig. 5. Longitudinal Motion of a Rod.

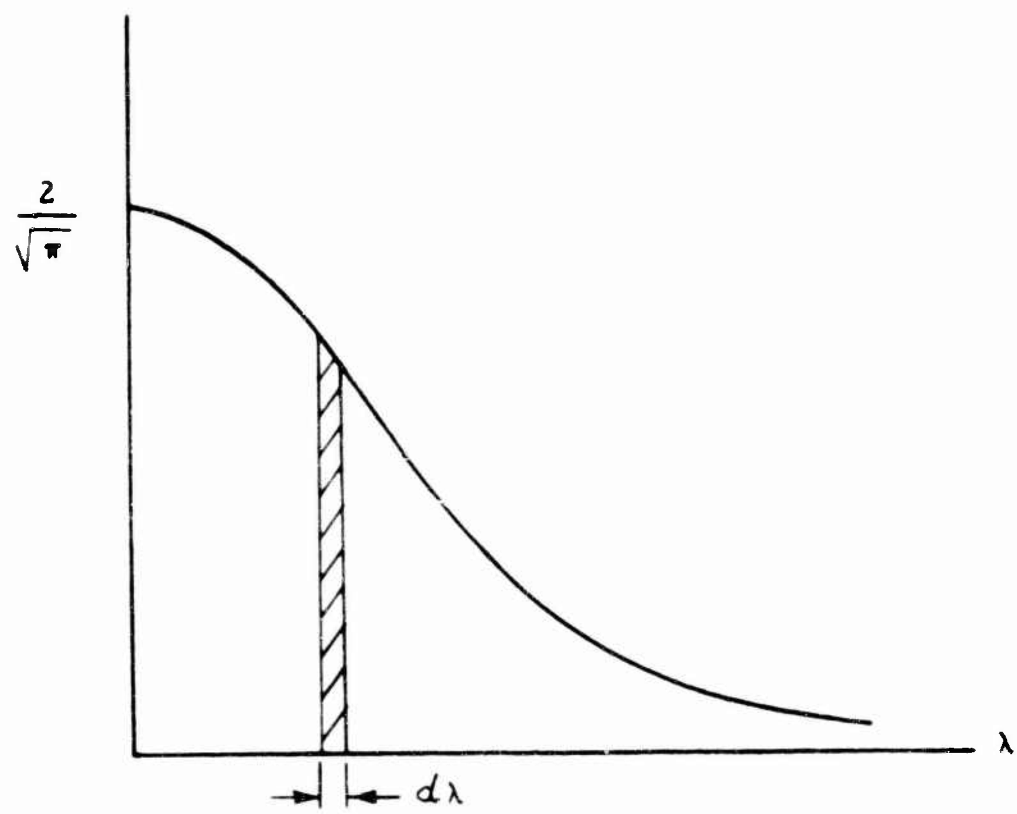

Fig. 6. Normal Distribution Curve.

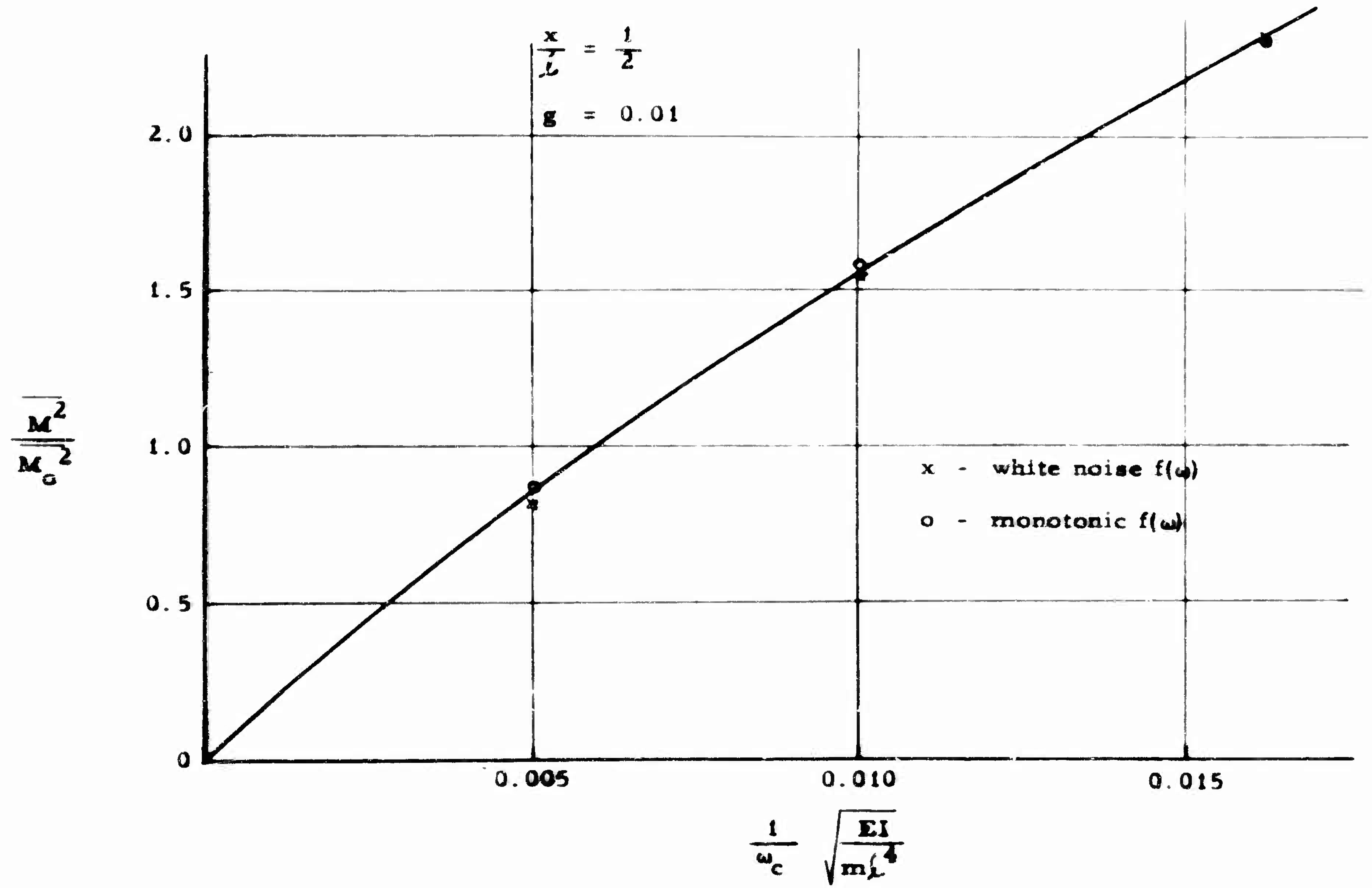

Figure 7. Flexural Motion of Beams.

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